

Quantum correction to the Kubo formula in closed mesoscopic systems

Mikhail A. Skvortsov

L. D. Landau Institute for Theoretical Physics, Moscow 117940, Russia

We study the energy dissipation rate in a mesoscopic system described by the parametrically-driven random-matrix Hamiltonian $H[\varphi(t)]$ for the case of linear bias $\varphi = vt$. Evolution of the field $\varphi(t)$ causes interlevel transitions leading to energy pumping, and also smears the discrete spectrum of the Hamiltonian. For sufficiently fast perturbation this smearing exceeds the mean level spacing and the dissipation rate is given by the Kubo formula. We calculate the quantum correction to the Kubo result that reveals the original discreteness of the energy spectrum. The first correction to the system viscosity scales $\propto v^{-2/3}$ in the orthogonal case and vanishes in the unitary case.

PACS numbers: 73.23.-b, 72.10.Bg, 03.65.-w

The Kubo formula [1] is one of the cornerstones of modern condensed matter physics. It is a standard tool for calculating various linear response functions, with conductivity as a prototypical example. Based on the lowest-order perturbation theory for a continuous spectrum, Kubo approach connects the kinetic response of a system with respect to some external field $\varphi(t)$ to the equilibrium correlator of generalized currents.

Application of the Kubo formula essentially relies on the assumption of a continuous spectrum [2]. (For the case of a discrete spectrum it gives either zero, when the frequency of the external field $\varphi(t)$ is out of resonance with any pair of energy levels, or infinity otherwise, indicating breakdown of the linear response theory.) The widespread usage of the Kubo formula for macroscopic objects is justified by the smallness of the mean level spacing $\Delta \propto \hbar^d/L^d$ (d is the dimensionality of space and L is the system size) which is usually smaller than the inelastic width Γ_{in} of energy levels. In an electron system, the inelastic smearing is due to interaction as well as escape to reservoirs operative for open systems.

For closed microscopic systems (e.g., quantum dots), the condition of continuous spectrum is violated at sufficiently low temperatures when the interaction-induced smearing Γ_{in} becomes smaller than Δ . In this case individual discrete levels are well resolved and the Kubo formula may fail to describe the dissipation in the system. Such a situation had recently been discussed in the context of vortex dynamics in layered superconductors at low temperatures [3, 4, 5, 6].

In a closed system with $\Gamma_{\text{in}} \ll \Delta$, the mechanism of dissipation depends on the rate $v = d\varphi/dt$ of variation of the external field $\varphi(t)$. For sufficiently slow perturbations with $v \ll v_K$ (the velocity v_K depends on the sensitivity of the spectrum to the change of φ and will be defined below), the system adiabatically follows the spectrum of its instantaneous Hamiltonian $H[\varphi(t)]$, and the dissipation is due to rare Landau-Zener transitions between individual levels [7]. Fast perturbations with $v \gg v_K$ cause multiple transitions between energy levels thereby transforming the discrete spectrum of the stationary Hamiltonian into a featureless quasi-continuous

spectrum, where the dissipation can be obtained with the help of the Kubo formula.

These two regimes had been studied by Wilkinson [8] who calculated the dissipation in the adiabatic ($v \ll v_K$) and Kubo ($v \gg v_K$) regimes assuming the Hamiltonian from the Wigner-Dyson random-matrix ensembles [9]. In the Kubo regime, the energy dissipation rate is given by the linear-response formula:

$$W_{\text{Kubo}} = \frac{\beta}{2} \pi C(0) v^2, \quad (1)$$

where $\beta = 1$ for the orthogonal (GOE) and 2 for the unitary (GUE) ensembles, and $C(0) \equiv \langle (\partial E_i / \partial \varphi)^2 \rangle / \Delta^2$ is the variance of the level velocity normalized by the mean level spacing Δ . (The system of units with $\hbar = 1$ is used throughout the Letter.) Equation (1) suggests that $C(0)$ has the meaning of the generalized conductance [10]. It determines the critical velocity $v_K \sim \Delta / \sqrt{C(0)}$ separating the adiabatic and Kubo regimes of dissipation. In the adiabatic limit, the dissipation rate nontrivially depends on the symmetry of the Hamiltonian [8]:

$$W_\beta = c_\beta v^{1+\beta/2}, \quad (2)$$

where $c_1 = (\pi/4)\Gamma(\frac{3}{4})[C(0)]^{3/4}\Delta^{1/2}$ and $c_2 = \pi C(0)$. Hence the dissipation is superohmic for the GOE, while for the GUE it remains ohmic, exactly coinciding with W_{Kubo} despite a very different mechanism of dissipation.

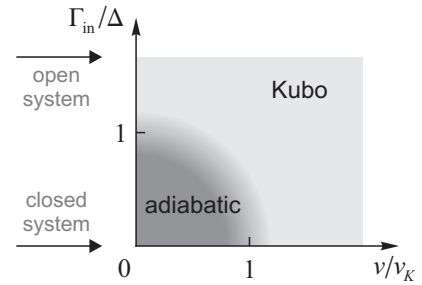


FIG. 1: Position of the adiabatic and Kubo regimes as a function of the inelastic width Γ_{in} and velocity v .

A schematic diagram indicating the regions of the adiabatic and Kubo regimes as a function of the inelastic width Γ_{in} and velocity v of external perturbation is shown in Fig. 1. Counterintuitively, the linear-response Kubo formula does not describe the low-velocity behavior of closed systems at $\Gamma_{\text{in}} \ll \Delta$!

In this Letter we address the question of how does the discreteness of energy levels of a stationary ($v = 0$) Hamiltonian manifest itself in the Kubo regime ($v \gg v_K$) when the levels are smeared into a quasi-continuous spectrum. We show that the relative correction to the high-velocity asymptotics (1) can be regularly expanded in integer powers of $(v_K/v)^{2/3}$. For the orthogonal symmetry, the first (one-loop) quantum correction to W_{Kubo} is given by

$$\frac{W_1}{W_{\text{Kubo}}} = 1 + \frac{\Gamma(\frac{1}{3})}{3^{2/3}} \left(\frac{v_K}{v}\right)^{2/3} + \dots, \quad (3)$$

where the omitted terms have the order of $O[(v_K/v)^{4/3}]$, and the crossover velocity is defined as $v_K \equiv (2^{1/2}/\pi^2)\Delta/\sqrt{C(0)}$. Thus, the remaining correlations in the quasi-continuous spectrum enhance dissipation at $v \gg v_K$, with a gradual crossover to the superohmic regime (2) at $v \ll v_K$. In the unitary case, the first and the second (two-loop) quantum correction to the Kubo result (1) vanish, making it tempting to conjecture that the identity $W_2 \equiv W_{\text{Kubo}}$ holds for all velocities.

Quantum corrections to the quasiclassical properties of disordered systems had been the subject of intense studies in the last decades [11, 12]. In treating these phenomena, the nonlinear σ -model was proven to be the most powerful tool both in the perturbative and nonperturbative regimes (where it often is the only possible approach). Three versions of the σ -model based on the supersymmetry [12], replica [13], and Keldysh [14, 15] techniques had been proposed for noninteracting systems.

The problem of energy pumping by the parametrically-driven Hamiltonian $H[\varphi(t)]$ belongs to the class of non-equilibrium problems, that dictates the choice of the Keldysh formalism as a solution tool. We will derive the Keldysh σ -model for the parametrically-driven random-matrix Hamiltonian and show that its saddle-point solution yields the kinetic equation for the distribution function, reproducing the Kubo result (1). Fluctuations near this saddle point are responsible for the quantum correction to the Kubo formula leading to Eq. (3).

We consider a time-dependent matrix Hamiltonian

$$H[\varphi(t)] = H_1 \cos k\varphi(t) + H_2 \sin k\varphi(t), \quad (4)$$

where H_1 and H_2 are the $N \times N$ matrices from the same GOE ($H^T = H$) distributed with the probability density $P(V) \propto \exp[-(\pi^2/2N\Delta^2)\text{tr} V^2]$, and $k = (\pi\sigma/\Delta)(2/N)^{1/2}$. The density of states for an instant Hamiltonian is given by the Wigner semicircle: $\rho(E) = \Delta^{-1}[1 - \pi^2 E^2/4N^2\Delta^2]^{1/2}$, with Δ being the mean level

spacing at the center of the band. The dispersion of the matrix elements

$$\left\langle \left| \frac{\partial H_{ij}}{\partial \varphi} \right|^2 \right\rangle = \sigma^2(1 + \delta_{ij}) \quad (5)$$

is independent of the matrix size N . The generalized conductance is then $C(0) = 2\sigma^2/\Delta^2$. The states of the Hamiltonian are filled by $N/2$ noninteracting fermions.

The time evolution governed by $\varphi(t)$ will change the state of the system and eventually pump some energy into it. The energy of the system will grow unless the inelastic interaction with the thermal-bath is taken into account. This interaction will establish a nonequilibrium steady state. Remarkably, however, that the energy dissipation rate is independent of the resulting distribution and hence can be calculated as a mean growth rate of the total system energy of noninteracting fermions [8].

Within the Keldysh formalism, the system is described by the action

$$S[\Psi] = -i \int_{-\infty}^{\infty} dt \Psi^\dagger(t) \left[i\tau_3 \frac{\partial}{\partial t} - H[\varphi(t)] \right] \sigma_3 \Psi(t), \quad (6)$$

where $\Psi(t)$ is a Grassmanian $4N$ -vector field acting in the direct product of the index space of the Hamiltonian, Keldysh (K) space and particle-hole (PH) space. The Pauli matrices in the K and PH spaces are denoted by σ_i and τ_i , respectively. The Keldysh forward-backward structure is a necessary ingredient of the dynamic formalism ensuring the correct normalization $Z \equiv \int D\Psi e^{-S} = 1$ in the absence of quantum sources, whereas the PH grading is introduced in order to handle the orthogonal symmetry of the Hamiltonian [12].

Derivation of the σ -model, which is a low-energy effective theory for the action (6), is a standard procedure. One has to average Z over the Hamiltonian (4), decouple it by the 4×4 matrix $Q_{tt'}$ and integrate over fermions Ψ . Keeping the terms which are finite in the limit $N \rightarrow \infty$ and assuming linear bias $\varphi = vt$ we arrive at the following action (the weight is e^{-S}) for the σ -model [16]:

$$S = \frac{\pi i}{2\Delta} \text{Tr} \hat{E} \tau_3 Q + \frac{\pi \Omega^3}{8\Delta} \int dt dt' (t - t')^2 \text{tr} Q_{tt'} Q_{t't}, \quad (7)$$

where $\Omega^3 \equiv \pi\sigma^2 v^2/\Delta = (\pi/2)C(0)v^2\Delta$. The first term in Eq. (7) is the standard random-matrix action [17] which is responsible for the whole spectral statistics. The second term is of kinetic origin; it accounts for interlevel transitions of the time-dependent Hamiltonian $H[\varphi(t)]$. The field theory with the action (7) describes both the adiabatic and Kubo regimes of dissipation on an equal footing. It is controlled by the single parameter

$$\frac{\Omega}{\Delta} = \frac{1}{\pi} \left(\frac{v}{v_K} \right)^{2/3} \quad (8)$$

which will be used hereafter as a measure of velocity v .

In the stationary case ($\Omega = 0$), the Keldysh Green function Q is diagonal in the energy representation [15]:

$$\Lambda = \begin{pmatrix} 1 & 2F \\ 0 & -1 \end{pmatrix} \otimes \tau_3, \quad (9)$$

where $F(E) = 1 - 2f(E)$, and $f(E)$ is the electron distribution function. The evolution of the distribution function at $\Omega \neq 0$ is described by the saddle point of the action (7). Varying with respect the constraint $Q^2 = \mathbb{1}$ one obtains the saddle-point equation $[Q, \delta S / \delta Q] = 0$. Seeking the solution in the form (9), we obtain the equation for the distribution function $f_{tt'}$:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'} \right) f_{tt'} = -\Omega^3 (t - t')^2 f_{tt'}. \quad (10)$$

The same equation had been obtained in Ref. 18 starting with the diagrammatic technique. Performing the Wigner transformation $f(E, t) = \int d\tau e^{iE\tau} f_{t+\tau/2, t-\tau/2}$ we arrive at the kinetic equation

$$\frac{\partial f(E, t)}{\partial t} = \Omega^3 \frac{\partial^2 f(E, t)}{\partial E^2}. \quad (11)$$

This is a diffusion equation in the energy space, with the bare “diffusion coefficient” $\mathcal{D}_0 = \Omega^3$. The rate of energy pumping for the system described by the kinetic equation (11) is given by

$$W = \int E \frac{\partial f(E, t)}{\partial t} \frac{dE}{\Delta} = \frac{\Omega^3}{\Delta}, \quad (12)$$

that coincides with the result of the Kubo formula (1).

The kinetic equation (11) is a true saddle-point of the action for all velocities v . The answer (12) for the dissipation rate predicted is valid, however, only in the Kubo regime, and is completely wrong in the adiabatic regime. The reason is that the saddle-point approximation is justified by the large value of the parameter Ω/Δ . In the adiabatic regime, the saddle-point approximation fails and one has to take all the Q -manifold into account. As a result of this procedure, the answer (2) should be reproduced. Note the interchange of steps with respect to the Wilkinson’s derivation [8]: He first calculates the probability of the Landau-Zener transition and then averages it over the distribution of avoided crossings. Here we, instead, first average over randomness and then extract the dissipation rate from the field theory (7). Thus it is a challenging problem to demonstrate how the adiabatic result (2) should be obtained from the field theory (7).

Quantum correction to the Kubo result (12) in the limit $\Omega/\Delta \gg 1$ (Δ/Ω is the loop expansion parameter) can be obtained in the regular way by expanding over Gaussian fluctuations near the saddle point. The matrix Q can be written as [15]

$$Q = U_F^{-1} P U_F, \quad U_F = \begin{pmatrix} 1 & F \\ 0 & -1 \end{pmatrix}, \quad (13)$$

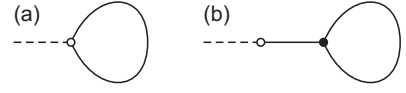


FIG. 2: 1-loop corrections to the system energy. The solid lines stand for the propagators (15) and (16), the dashed line denotes for the source field $\kappa(t)$, and the open and solid vertices come from the terms S_κ and S , respectively.

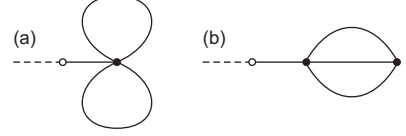


FIG. 3: 2-loop corrections to the system energy in the unitary case.

with the Hermitian matrix P having an additional symmetry $P^T = \sigma_1 \tau_2 P \tau_2 \sigma_1$ imposed by the orthogonal symmetry of the Hamiltonian. We choose the standard rational parameterization of the P -matrix, $P = \sigma_3 \tau_3 (1 + V/2)(1 - V/2)^{-1}$, with the unit Jacobian. The matrix V is explicitly given by

$$V = \begin{pmatrix} 0 & a & b & 0 \\ -a^\dagger & 0 & 0 & -b^T \\ -b^\dagger & 0 & 0 & a^T \\ 0 & b^* & -a^* & 0 \end{pmatrix}, \quad (14)$$

where the inner (outer) grading corresponds to the PH (K) space. The elements $a_{tt'}$ and $b_{tt'}$ describe cooperon and diffuson modes, respectively. Their bare propagators read [18, 19, 20, 21]

$$\langle a_{t_1 t_2} a_{t_3 t_4}^* \rangle = \frac{2\Delta}{\pi} \delta(t_{13} + t_{24}) \theta(t_{13}) e^{-\Omega^3 (t_{13}^3/3 + t_{13} t_{14}^2)}, \quad (15)$$

$$\langle b_{t_1 t_2} b_{t_3 t_4}^* \rangle = \frac{2\Delta}{\pi} \delta(t_{13} - t_{24}) \theta(t_{13}) e^{-\Omega^3 t_{13} t_{12}^2}, \quad (16)$$

where $t_{ij} \equiv t_i - t_j$. In the stationary case ($\Omega = 0$), Eqs. (15) and (16) describe the standard cooperon and diffuson in the time domain. The exponential decay of the correlators at the time scale Ω^{-1} is a manifestation of dephasing by the time-dependent perturbation [18, 19].

The system energy $\langle E \rangle$ (apart from an additive constant) can then be obtained as a functional derivative with respect to the quantum source field $\kappa(t)$:

$$\langle E(t) \rangle = \frac{1}{2} \frac{\delta Z[\kappa]}{\delta \kappa(t)} \Big|_{\kappa=0}, \quad Z[\kappa] = \int e^{-S - S_\kappa} DQ, \quad (17)$$

where the source action $S_\kappa = (\pi/2\Delta) \text{Tr} \kappa \hat{E} \sigma_1 \tau_3 Q$. Indeed, the presence of the Pauli matrix σ_1 cuts the off-diagonal in the Keldysh space elements of the matrix Q , thus projecting onto the upper right (Keldysh) block containing the distribution function renormalized by quantum fluctuations (the lower left block vanishes due to causality).

At the saddle point, Eq. (17) reproduces the result (12). The one-loop diagrams are shown in Fig. 2. The diagram (a) coming from the expansion of the source term S_κ contains either $\langle a(t_1, t_2) a^*(t_1, t_3) \rangle$ or $\langle b(t_1, t_2) b^*(t_1, t_3) \rangle$ which are proportional to $\theta(0)$. In the Keldysh formalism, the Heaviside θ -function of zero argument evaluates to zero, that is related with the causality of the theory [17]. Calculating the diagram (b) we obtain for the quantum correction to the dissipation rate:

$$\frac{\delta W_1}{W_{\text{Kubo}}} = \frac{\Gamma(\frac{1}{3})}{3^{2/3}\pi} \frac{\Delta}{\Omega}, \quad (18)$$

which after rewriting in terms of velocity leads to Eq. (3). This result may be interpreted in terms of renormalization of the bare “diffusion coefficient” in Eq. (11): $\mathcal{D}_0 \rightarrow \mathcal{D}_0(1 + \delta W_1/W_{\text{Kubo}})$, bearing a natural analogy with the weak localization phenomena. In our case, the ratio Ω/Δ plays the role of the dimensionless conductance as it controls the expansion in terms of the diffusive modes.

In the unitary case, the diagram of Fig. 2(b) vanishes indicating the absence of the one-loop quantum correction to the Kubo result. In the two-loop approximation, only the diagrams shown in Fig. 3 contribute to the dissipation rate for the GUE. Each of them has the order of $(\Delta/\Omega)^2$, but their sum is zero. Thus, for the case of the unitary symmetry, the two-loop correction also vanishes. Taking into account the coincidence between the low- and high-velocity asymptotics (1) and (2) for $\beta = 2$, such a cancelation is a strong argument in favor of the exact identity $W_2 = W_{\text{Kubo}}$ valid for all velocities. At present we cannot prove this conjecture but we hope that this can be done by an accurate analysis of the σ -model (7). We conjecture that the independence of the dissipation on the velocity v might be another manifestation of the peculiar properties of the unitary group [22]. It is worth mentioning that the direct quantum-mechanical calculation of the transition rates at $v \sim v_K$ prior to disorder averaging seems completely impossible. Therefore, the identity $W_2 = W_{\text{Kubo}}$ for the averaged dissipation rate would be a highly nontrivial fact.

The results obtained may be relevant for the description of heating effect in closed quantum dots whose shape is being changed by the low-frequency external perturbation. Vortex motion in impure superconductors is another field of application, where the conjecture $W_2 = W_{\text{Kubo}}$ would indicate the independence of the dissipative flux-flow conductivity on the velocity of vortex motion [3, 6].

Summarizing, we have developed the Keldysh σ -model approach to study energy pumping in the parametrically-driven random-matrix ensembles. We calculated the leading quantum correction to the high-velocity dissipation, which reveals the original discreteness of the spectrum of the stationary Hamiltonian. This correction

emerges in the one-loop approximation for the GOE, and is absent within the two-loop accuracy for the GUE.

I am grateful to D. M. Basko, Ya. M. Blanter, D. A. Ivanov, M. V. Feigel'man, V. E. Kravtsov, A. I. Larkin, and Yu. V. Nazarov for illuminating discussions. Financial support from the SCOPES program of Switzerland, the Dutch Organization for Fundamental Research (NWO), the Russian Foundation for Basic Research under grants 01-02-17759 and 02-02-06238, the program “Quantum Macrophysics” of the Russian Academy of Sciences, the Russian Ministry of Science, and the Russian Science Support Foundation is acknowledged.

-
- [1] R. Kubo, Can. J. Phys. **34**, 1274 (1956).
 - [2] G. D. Mahan, *Many-particle physics*, 3rd ed. (Kluwer, Boston, 2000).
 - [3] M. V. Feigel'man and M. A. Skvortsov, Phys. Rev. Lett. **78**, 2640 (1997).
 - [4] A. I. Larkin and Yu. N. Ovchinnikov, Phys. Rev. B **57**, 5457 (1998).
 - [5] A. A. Koulakov and A. I. Larkin, Phys. Rev. B **60**, 14597 (1999).
 - [6] M. A. Skvortsov, D. A. Ivanov, and J. Blatter, preprint cond-mat/0207500.
 - [7] L. D. Landau, Phys. Z. Sowjetunion **2**, 46 (1932); C. Zener, Proc. R. Soc. London, Ser. A **137**, 696 (1932).
 - [8] M. Wilkinson, J. Phys. A **21**, 4021 (1988).
 - [9] M. L. Mehta, *Random Matrices* (Academic Press, Boston, 1991).
 - [10] B. D. Simons and B. L. Altshuler, Phys. Rev. B **48**, 5422 (1993).
 - [11] L. P. Gor'kov, A. I. Larkin, D. E. Khmel'nitskii, Pis'ma v ZhETF **30**, 248 (1979) [JETP Lett. **30**, 228 (1979)].
 - [12] K. B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, New York, 1997).
 - [13] F. J. Wegner, Z. Phys. B **35**, 207 (1979); K. B. Efetov, A. I. Larkin, and D. E. Khmel'nitskii, Sov. Phys. JETP **52**, 568 (1980).
 - [14] M. L. Horbach and G. Schön, Ann. Phys. **2**, 51 (1993).
 - [15] A. Kamenev and A. Andreev, Phys. Rev. B **60**, 2218 (1999).
 - [16] The general form of the kinetic term valid for an arbitrary dependence of $\varphi(t)$ is $S_{\text{kin}} = (\beta\pi^2 C(0)/16) \int dt dt' [\varphi(t) - \varphi(t')]^2 \text{tr} Q_{tt'} Q_{t't}$. In this form it is valid both for the GOE and GUE.
 - [17] A. Altland and A. Kamenev, Phys. Rev. Lett. **85**, 5615 (2000).
 - [18] V. I. Yudson, E. Kanzieper, and V. E. Kravtsov, Phys. Rev. B **64**, 045310 (2001).
 - [19] M. G. Vavilov and I. L. Aleiner, Phys. Rev. B **60**, R16311 (1999).
 - [20] X.-B. Wang and V. E. Kravtsov, Phys. Rev. B **64**, 033313 (2001).
 - [21] There is also a finite average $\langle b_{t_1 t_2} b_{t_3 t_4} \rangle$ but it is irrelevant for the calculation of the mean dissipation rate.
 - [22] M. R. Zirnbauer, cond-mat/9903338.